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# SPLINE SYSTEMS AS BASES IN HARDY SPACES

BY

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ABSTRACT

It is known that the spline system of order $m$ is an unconditional basis for $H^p[0,1]$ when $p > 1/(m+2)$ and a Schauder basis when $p \geqq 1/(m+2)$. We show that these results are sharp.

## 1. Introduction

Set $I = [0,1]$ and let $(f_n^{(0)})_{n=0}^{\infty}$ denote the Franklin system and $(f_n^{(m)})_{n=-m}^{\infty}$, $m \geqq 1$, spline systems of higher order on $I$ (for a definition see e.g. Z. Ciesielski [1]).

We shall write $f_n$ instead of $f_n^{(m)}$ and set $f_n(t) = 0$ for $t \in \mathbf{R} \setminus I$. We let $H^p = H^p(\mathbf{R})$, $0 < p < \infty$, denote the usual Hardy spaces on $\mathbf{R}$ (cf. [2]). For $\alpha > 0$ we set $N = [\alpha]$, where [ ] denotes the integral part, and $\delta = \alpha - N$. If $\alpha$ is not an integer set

$$\dot{\Lambda}_\alpha = \left\{ \varphi \in C^N(\mathbf{R});\ \sup_{h \neq 0} \|\Delta_h D^N \varphi\|_\infty / |h|^\delta < \infty \right\}$$

(here $\Delta_h F(x) = F(x+h) - F(x)$) and if $\alpha$ is an integer set

$$\dot{\Lambda}_\alpha = \left\{ \varphi \in C^{N-1}(\mathbf{R});\ \sup_{h \neq 0} \|\Delta_h^2 D^{N-1} \varphi\|_\infty / |h| < \infty \right\}.$$

Also set $\tilde{\Lambda}_\alpha = \dot{\Lambda}_\alpha / P^N$, where $P^N$ denotes the class of polynomials of degree $\leqq N$. The projection from $\dot{\Lambda}_\alpha$ to $\tilde{\Lambda}_\alpha$ is denoted $\pi$. For $0 < p \leqq 1$ set $\alpha = 1/p - 1$. It is then well-known that for $0 < p < 1$, $\tilde{\Lambda}_\alpha$ is the dual space of $H^p$. If $\varphi \in \dot{\Lambda}_\alpha$ set $\varphi(f) = (\pi(\varphi))(f)$ for $f \in H^p$.

Also set $H^p(I) = \{f \in H^p(\mathbf{R});\ \operatorname{supp} f \subset I$ and $\varphi(f) \in \mathbf{R}$ for every real-valued $\varphi \in \dot{\Lambda}_\alpha\}$, $0 < p < 1$.

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It is also well-known that $(H^1)^* = \mathrm{BMO}$ and we set $H^1(I) = \{f \in H^1(\mathbf{R});$ $\operatorname{supp} f \subset I$ and $f$ real-valued$\}$.

The following theorem was proved in [6].

THEOREM A. *Assume $m \geqq 0$. Then $(f_n)_{-m+N+1}^{\infty}$ is an unconditional basis of $H^p(I)$ if $1/(m+2) < p \leqq 1$ (here $N = [1/p - 1]$).*

P. Oswald [4] has obtained the following result.

THEOREM B. *Assume $m \geqq 0$. Then $(f_n)_2^{\infty}$ is a basis of $H^p(I)$ if $p = 1/(m+2)$.*

The main purpose of this paper is to prove that these results are sharp. We shall prove the following theorems.

THEOREM 1. *Assume $p < 1/(m+2)$ and $k \geqq -m$. Then $(f_n)_k^{\infty}$ is not a basis of $H^p(I)$.*

THEOREM 2. *$(f_n)_2^{\infty}$ is not an unconditional basis of $H^p(I)$ if $p = 1/(m+2)$.*

We remark that the above theorems hold also if we replace $H^p(I)$ by the space $\{f \in H^p(\mathbf{R});\ \operatorname{supp} f \subset I\}$.

Throughout the paper we use the notation $\alpha = 1/p - 1$ and $N = [\alpha]$, where $0 < p \leqq 1$.

If $n \geqq 2$ we write $n = 2^j + l$ where $j \geqq 0$, $1 \leqq l \leqq 2^j$, and set $t_n = (l - \frac{1}{2})2^{-j}$ and $I_n = [(l-1)2^{-j}, l2^{-j}]$.

We also remark that it follows from the definition of $H^p(\mathbf{R})$ that if $f \in H^p \cap C_0^{\infty}$ then

$$\int f(t)t^k dt = 0, \qquad k = 0, 1, 2, \cdots, N.$$

## 2. Proof of Theorem 1

PROOF OF THEOREM 1. We first assume that $0 < p \leqq 1/(m+3)$ and that $(f_n)_k^{\infty}$ is a basis of $H^p(I)$, where $k \geqq -m$. It follows that $f_n \in H^p(I)$, $n \geqq k$. We have $1/p \geqq m+3$ and hence $N = [1/p - 1] \geqq m+2$ and $\int f_n(t)t^{m+2}dt = 0$, $n \geqq k$. $(f_n)_{-m}^{\infty}$ is a complete system in $L^2(I)$ and expanding the function $t^{m+2}\chi_I$ in this system we obtain

$$t^{m+2}\chi_I = \sum_{n=-m}^{k-1} c_n f_n$$

(where $\chi_I$ denotes the characteristic function of $I$). The right-hand side is a piecewise polynomial of degree $\leqq m+1$ while the left-hand side is of degree $m+2$ in $I$. Hence we obtain a contradiction.

Now assume that $1/(m+3) < p < 1/(m+2)$ and that $(f_n)_k^\infty$ is a basis of $H^p(I)$. First assume $k \leqq 1$. Since $f_1$ is a polynomial of degree $m+1$ and $N = m+1$ we obtain $(f_1, f_1) = 0$ which gives a contradiction. We conclude that $k \geqq 2$.

For every $f \in H^p(I)$ there exist unique coefficients $a_n = a_n(f)$ such that $f = \Sigma_k^\infty a_n(f) f_n$ with convergence in $H^p$. We first claim that

$$(1) \qquad |a_n(f)| \leqq C_n \|f\|_{H^p}, \qquad n \geqq k.$$

$H^p(I)$ is a complete metric space with the metric $d(f,g) = \|f-g\|_{H^p}^p$. We set

$$\|f\|_0 = \sup_n \left\| \sum_{i=k}^n a_i f_i \right\|_{H^p}, \qquad f \in H^p(I),$$

and

$$d_0(f,g) = \|f-g\|_0^p, \qquad f, g \in H^p(I).$$

It is then clear that

$$(2) \qquad |a_n(f)| \leqq C_n \|f\|_0, \qquad n \geqq k.$$

$d_0$ is a metric on $H^p(I)$ and we shall prove that it is complete. Therefore assume that $(g^n)_1^\infty$ is a sequence in $H^p(I)$ such that $\|g^n - g^m\|_0 \to 0$ as $n, m \to \infty$. It follows that there exists a sequence $(a_i)_k^\infty$ of real numbers such that $\lim_{n\to\infty} a_i(g^n) = a_i$, $i \geqq k$.

We shall prove that $\Sigma_k^\infty a_i f_i$ converges in $H^p$ with sum $g$ and that $\lim_{n\to\infty} \|g^n - g\|_0 = 0$.

Choose $\varepsilon > 0$. Then there exists $M$ such that $\|g^n - g^m\|_0 < \varepsilon$ if $n, m \geqq M$, i.e.

$$\left\| \sum_k^l (a_i(g^n) - a_i(g^m)) f_i \right\|_{H^p}^p < \varepsilon^p, \qquad l = k, k+1, k+2, \cdots; \quad n, m \geqq M.$$

Hence

$$(3) \qquad \left\| \sum_k^l (a_i(g^n) - a_i) f_i \right\|_{H^p}^p \leqq \varepsilon^p, \qquad l \geqq k, \quad n \geqq M.$$

We have

$$\left\| \sum_{l_1}^{l_2} a_i f_i \right\|_{H^p}^p \leqq \left\| \sum_{l_1}^{l_2} (a_i(g^M) - a_i) f_i \right\|_{H^p}^p + \left\| \sum_{l_1}^{l_2} a_i(g^M) f_i \right\|_{H^p}^p \leqq 2\varepsilon^p + \varepsilon,$$

if $l_1$ and $l_2$ are large enough. Hence $\Sigma_k^\infty a_i f_i$ converges in $H^p$ and letting $g$ denote this sum we conclude from (3) that $\|g^n - g\|_0 \to 0$ as $n \to \infty$. It follows that $d_0$ is a complete metric.

We have $\|f\|_{H^p} \leqq \|f\|_0$ and it follows from the open mapping theorem (see W. Rudin [5], pp. 48–49) that $\|f\|_0 \leqq C \|f\|_{H^p}$. Hence (1) is a consequence of (2).

Let $b$ be a $p$-atom with support on $I$, i.e. $\operatorname{supp} b \subset J \subset I$, where $J$ is an interval, $\|b\|_\infty \leqq |J|^{-1/p}$ and $\int b(x)x^l dx = 0$, $l = 0, 1, \cdots, N$. We have $b = \Sigma_2^\infty c_n f_n$ with convergence in $L^2$, where $c_n = \int bf_n dt$. It follows that also $b = \Sigma_2^\infty c_n f_n$ with convergence in $H^p$ (cf. the proofs of lemmas 3 and 4 in [6]). However, we also have $b = \Sigma_k^\infty a_n(b) f_n$ with convergence in $H^p$.

First assume $k = 2$. It then follows that

$$a_n(b) = c_n = \int bf_n dt. \tag{4}$$

We shall now make a special choice of $b$. Choose $\varphi \in C_0^\infty(\mathbf{R})$ such that $\varphi$ is odd, $\operatorname{supp} \varphi \subset (-1/2, 1/2)$, $\varphi(x) \geqq 0$ for $x \leqq 0$ and $\varphi(x) \leqq 0$ for $x \geqq 0$, $|D^{m+1}\varphi| \leqq 1$ and $\varphi \not\equiv 0$. Set

$$\varphi_a(x) = a^{m+1-1/p}\varphi(x/a), \qquad a > 0.$$

Now fix $n \geqq 2$. Set $B_a(x) = \varphi_a(x - t_n)$ and $b_a = D^{m+1}B_a$. It is then easy to see that $b_a$ is a $p$-atom with support in the interval $J = (t_n - a/2, t_n + a/2)$. We have

$$a_n(b_a) = \int b_a f_n dt = (-1)^{m+1} \int B_a D^{m+1} f_n dt.$$

We now invoke the fact that $D^{m+1} f_n$ has a jump with absolute value $d_n > 0$ at $t_n$ (cf. the proof of lemma 2 in [6]). It follows that

$$|a_n(b_a)| \geqq ca^{m+1-1/p} d_n a$$

if $a$ is small and positive, where $c > 0$.

From (1) it follows that $|a_n(b_a)| \leqq C_n$ and hence $a^{m+2-1/p} \leqq C_n$. This can only hold if $m + 2 - 1/p \geqq 0$, i.e. $p \geqq 1(m+2)$. We have obtained a contradiction and the proof is complete in the case $k = 2$.

Now assume $k \geqq 3$. We have

$$f_2 = \sum_k^\infty a_n(f_2) f_n,$$
$$\vdots$$
$$f_{k-1} = \sum_k^\infty a_n(f_{k-1}) f_n.$$

If $b$ is a $p$-atom we therefore obtain

$$\begin{aligned} b = \sum_2^\infty c_n f_n &= c_2 \sum_k^\infty a_n(f_2) f_n + \cdots + c_{k-1} \sum_k^\infty a_n(f_{k-1}) f_n + \sum_k^\infty c_n f_n \\ &= \sum_k^\infty (a_n(f_2)c_2 + \cdots + a_n(f_{k-1})c_{k-1} + c_n) f_n \end{aligned}$$

with convergence in $H^p$, where $c_n = \int bf_n dt$. It follows that

$$a_n(b) = a_n(f_2)c_2 + \cdots + a_n(f_{k-1})c_{k-1} + c_n, \qquad n \geqq k. \tag{5}$$

We fix $n \geqq k$ and choose $b_a$ as above. It is clear that $|a_n(b_a)| \leqq C_n$ but the above estimates show that $|c_n| \to \infty$ as $a \to 0$ (here $c_i = c_i(b_a) = \int b_a f_i dt$). If we also observe that $D^{m+1} f_i$ is constant in a neighbourhood of $t_n$, $2 \leqq i \leqq k-1$, we obtain $\lim_{a\to 0} c_i(b_a) = 0$, $2 \leqq i \leqq k-1$, and hence (5) yields a contradiction. The proof is complete.

## 3. Proof of Theorem 2

We first prove the following lemma.

LEMMA 1. *Assume* $c_i \in \mathbf{R}$, $i = 0, 1, \cdots, m$. *Then there exists* $f \in C^m([0, m+1])$ *with the following properties*:

$$f \text{ is a polynomial of degree} \leqq m+1 \text{ in every interval } [k, k+1], \quad k = 0, 1, \cdots, m, \tag{6}$$

$$f^{(k)}(0) = c_k, \qquad k = 0, 1, \cdots, m, \tag{7}$$

$$f^{(k)}(m+1) = 0, \qquad k = 0, 1, \cdots, m, \tag{8}$$

$$\left(\int_0^{m+1} |f|^2 dx\right)^{1/2} \leqq C \sup_i |c_i|, \qquad \text{where } C \text{ depends only on } m. \tag{9}$$

PROOF. We set

$$f(x) = \sum_0^m \frac{c_j}{j!} x^j + \sum_0^m d_j (x-j)_+^{m+1},$$

where we shall determine $(d_j)_0^m$ so that (8) holds. For $m < x \leqq m+1$ we have

$$f(x) = \sum_0^m \frac{c_j}{j!} x^j + h(x),$$

where $h(x) = \sum_0^m d_j (x-j)^{m+1}$. It follows that

$$h'(x) = (m+1) \sum_0^m d_j (x-j)^m,$$

$$h''(x) = (m+1)m \sum_0^m d_j (x-j)^{m-1},$$

$$\vdots$$

$$h^{(m)}(x) = (m+1)! \sum_0^m d_j (x-j)$$

for $m < x \leqq m+1$.

Since the determinant

$$\begin{vmatrix} (m+1)^{m+1} & m^{m+1} & \cdots & 2^{m+1} & 1 \\ (m+1)^m & m^m & & 2^m & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ m+1 & m & & 2 & 1 \end{vmatrix}$$

does not vanish, there is exactly one choice of $(d_j)_0^m$ such that (8) holds. It also follows that $\sup_i |d_i| \leqq C \sup_i |c_i|$ and (9) follows from this estimate.

We shall now introduce some notation which has been used in J.-O. Strömberg [7]. Let $A_0 = \mathbf{Z}_+ \cup \{0\} \cup \frac{1}{2}\mathbf{Z}_-$ and $A_1 = A_0 \cup \{\frac{1}{2}\}$. $A_0$ splits $\mathbf{R}$ into intervals $(I_\sigma)_{\sigma \in A_0}$, where $\sigma$ is the left endpoint of $I_\sigma$. Let $S_0^m$ be the subspace of functions $f$ in $L^2(\mathbf{R})$ such that $f \in C^m(\mathbf{R})$ and is a real polynomial of degree $\leqq m+1$ on each $I_\sigma$, $\sigma \in A_0$. Let $S_1^m$ be the corresponding subspace of $L^2(\mathbf{R})$ with the set $A_0$ replaced by $A_1$. Then there is a function $\tau \in L^2(\mathbf{R})$ which is uniquely defined modulo sign by the conditions

$$\tau \in S_1^m, \tag{10}$$

$$\tau \perp S_0^m, \quad \text{i.e.,} \quad \int_{\mathbf{R}} \tau(x) f(x) dx = 0 \qquad \text{for all } f \in S_0^m, \tag{11}$$

$$\|\tau\|_2 = 1. \tag{12}$$

Now assume $n \geqq 2$, $n = 2^j + l$. Set $F_n(t) = 2^{-j/2} f_n(2^{-j}(t + l - 1))$, $t \in \mathbf{R}$. Thus $\|F_n\|_2 = 1$ and $F_n$ is obtained by a dilation and translation of $f_n$ which maps the interval $I_n = [(l-1)2^{-j}, l2^{-j}]$ on $I$ and $t_n$ on the point $\frac{1}{2}$. We have the following theorem.

THEOREM 3. *If $\delta > 0$ then*

$$\lim_{\substack{n \to \infty \\ \delta \leqq t_n \leqq 1-\delta}} F_n = \tau$$

*with convergence in $L^2(\mathbf{R})$ (if the sign of $f_n$ is chosen correctly).*

REMARK. The condition $\delta \leqq t_n \leqq 1 - \delta$ can be replaced by the condition $\delta_n \leqq t_n \leqq 1 - \delta_n$, where $n\delta_n \to \infty$.

PROOF OF THEOREM 3. Set $b = 2^j - l + 1$ and $a = -l + 1$ so that $F_n$ vanishes outside the interval $[a, b]$. We then choose $\varepsilon_n \in L^2(\mathbf{R})$ such that $\varepsilon_n = 0$ outside the intervals $J_1 = [b, b + m + 1]$ and $J_0 = [a - (m+1), a]$ and so that: $\varepsilon_n \in C^m$ outside the points $b$ and $a$; $\varepsilon_n$ is a polynomial of degree $\leqq m+1$ in each interval

$[k, k+1]$, $k \in \mathbf{Z}$, if $[k, k+1] \subset J_1$ or $J_0$; the right-hand derivatives of $\varepsilon_n$ of order $\leqq m$ are equal to the corresponding left-hand derivatives of $F_n$ at the point $b$, the left-hand derivatives of $\varepsilon_n$ of order $\leqq m$ are equal to the corresponding right-hand derivatives of $F_n$ at the point $a$, $\| \varepsilon_n \|_2 \leqq Cr^n$, $0 < r < 1$.

The existence of such a function $\varepsilon_n$ is a consequence of Lemma 1, the exponential decay of $D^k f_n$ (cf. Z. Ciesielski [1], theorem 6.1) and the assumption $\delta \leqq t_n \leqq 1 - \delta$.

The definition of $f_n$ implies that $F_n \perp S_0^m$ and from the definition of $\varepsilon_n$ it follows that $F_n + \varepsilon_n \in S_1^m$. Let $P_0$ denote the orthogonal projection on $S_0^m$. Then

$$F_n + \varepsilon_n = P_0(F_n + \varepsilon_n) + (F_n + \varepsilon_n, \tau)\tau$$

and

$$\| F_n + \varepsilon_n \|_2^2 = \| P_0(F_n + \varepsilon_n) \|_2^2 + (F_n + \varepsilon_n, \tau)^2.$$

Since $P_0 F_n = 0$ and $\lim_{n \to \infty} \| \varepsilon_n \|_2 = 0$ it follows that

$$\| F_n \|_2^2 - (F_n, \tau)^2$$

tends to zero as $n$ tends to infinity. But $\| F_n \|_2 = 1$ and hence $\lim_{n \to \infty} (F_n, \tau)^2 = 1$. Choosing the sign of $f_n$ correctly we obtain $\lim_{n \to \infty} (F_n, \tau) = 1$. It follows that

$$\| F_n - \tau \|_2^2 = \| F_n \|_2^2 + \| \tau \|_2^2 - 2(F_n, \tau) = 2 - 2(F_n, \tau) \to 0 \quad \text{as } n \to \infty$$

and the theorem is proved.

We shall finally prove Theorem 2.

Proof of Theorem 2. We first prove that $D^{m+1}\tau$ has a jump at some point $k > 0$, where $k$ is an integer. Assume that $D^{m+1}\tau$ has no jump at the points $k = 1, 2, 3, \cdots$. Then $\tau$ is a polynomial on the interval $[\frac{1}{2}, \infty)$ and belongs to $L^2$ and hence $\tau$ vanishes on $[\frac{1}{2}, \infty)$. Since $D^{m+1}\tau$ has a jump at the point $\frac{1}{2}$ we have

$$\tau(x) = c_0(\tfrac{1}{2} - x)_+^{m+1}, \qquad 0 \leqq x \leqq 1,$$

where $c_0 \neq 0$. Now choose $g \in S_0^m$ with support in $[0, \infty)$ such that $g(x) = x^{m+1}$, $0 \leqq x \leqq 1$. Then

$$\int_{\mathbf{R}} g\tau dx = \int_0^{1/2} c_0(\tfrac{1}{2} - x)^{m+1} x^{m+1} dx \neq 0,$$

which contradicts the fact that $\tau \perp S_0^m$.

We have proved that there exists a point $k > 0$ such that $D^{m+1}\tau$ has a jump at this point. It then follows from Theorem 3 that $D^{m+1}F_n$ has a jump of absolute value $\geqq c > 0$ at the point $k$ if $n$ is large enough and $\delta \leqq t_n \leqq 1 - \delta$. Hence

$D^{m+1}f_n$ has a jump with absolute value $\geqq c2^{j(m+3/2)}$ at the point $2^{-j}(k+l-1)$ (here $n=2^j+l$). We choose $l_j=2^{j-1}-k+1$ and set $n_j=2^j+l_j$ and conclude that $D^{m+1}f_{n_j}$ has a jump of absolute value $\geqq c2^{j(m+3/2)}$ at the point $\frac{1}{2}$ for $j \geqq j_0$. Now assume that $p=1/(m+2)$ and that $(f_n)_2^\infty$ is an unconditional basis of $H^p(I)$. Then for each $f \in H^p(I)$ we have $f=\Sigma_2^\infty a_n f_n$ with convergence in $H^p$, where $a_n = a_n(f)$, and every rearrangement of this series also converges.

First we claim that

$$\Big\| \sum_{n\in\sigma} a_n f_n \Big\|_{H^p} \leqq C_f$$

for every finite set $\sigma$ of integers $\geqq 2$, where the constant $C_f$ does not depend on $\sigma$. We shall prove that if the claim is not true then there exists an enumeration $(n_j)_1^\infty$ of the set $\{2,3,4,\cdots\}$ such that $\Sigma_1^\infty a_{n_j} f_{n_j}$ diverges in the $H^p$ norm.

To do this first set $N_0=0$. Assume $N_{j-1}$ is defined. If the claim does not hold then there is a finite set $\sigma_j$ of integers $\geqq 2$ such that

$$\Big\| \sum_{n\in\sigma_j} a_n f_n \Big\|_{H^p}^p \geqq N_{j-1}+j.$$

Let $\bar{\sigma}_j=\{2,3,\cdots,\max\sigma_j\}$ and

$$N_j = \max_{\sigma\subset\bar{\sigma}_j} \Big\| \sum_{n\in\sigma} a_n f_n \Big\|_{H^p}^p.$$

Finally set $\tilde{\sigma}_j=\bar{\sigma}_{j-1}\cup\sigma_j$, where $\bar{\sigma}_0=\varnothing$. Then $\bar{\sigma}_0\subset\bar{\sigma}_1\subset\bar{\sigma}_2\subset\cdots$, $\tilde{\sigma}_1\subset\tilde{\sigma}_2\subset\tilde{\sigma}_3\subset\cdots$ and $\bigcup_1^\infty\tilde{\sigma}_j=\{2,3,4,\cdots\}$. Furthermore

$$\Big\| \sum_{\tilde{\sigma}_j} a_n f_n \Big\|_{H^p}^p \geqq \Big\| \sum_{\sigma_j} a_n f_n \Big\|_{H^p}^p - \Big\| \sum_{\tilde{\sigma}_j\setminus\sigma_j} a_n f_n \Big\|_{H^p}^p \geqq (N_{j-1}+j)-N_{j-1}=j.$$

Now we choose the integers $(n_k)_1^\infty$ in the following way. First we take the integers in $\tilde{\sigma}_1$ in any order, then the integers in $\tilde{\sigma}_2\setminus\tilde{\sigma}_1$, and so on. In this way $\{n_1,n_2,\cdots,n_k\}=\tilde{\sigma}_j$ for some $k_j$ and the divergence of $\Sigma_1^\infty a_{n_k}f_{n_k}$ is obvious.

This completes the proof of the claim.

Let $\theta=(\theta_n)_2^\infty$ be any sequence of numbers $-1$, 0, 1 of which all but finitely many are 0. It follows that $M_\theta f=\Sigma_2^\infty \theta_n a_n f_n$ satisfies $\sup_\theta \|M_\theta f\|_{H^p}<\infty$ for each $f\in H^p(I)$ and since $|a_n|=|a_n(f)|\leqq C_n\|f\|_{H^p}$, we also have

$$\|M_\theta f\|_{H^p} \leqq C_\theta \|f\|_{H^p}.$$

By the Banach–Steinhaus theorem ([5], p. 44, theorem 2.6) we conclude that

$$\|M_\theta f\|_{H^p} \leqq C\|f\|_{H^p}$$

with the constant $C$ independent of $\theta$ and $f$ (cf. also [3]).

Using a property of the Rademacher functions (see A. Zygmund [8], p. 213) we then find that

$$\left\| \left( \sum_{2}^{\infty} a_n^2 f_n^2 \right)^{1/2} \right\|_p \leqq C \|f\|_{H^p}, \qquad f \in H^p(I). \tag{13}$$

We choose $\varphi$ and $\varphi_a$ as in the proof of Theorem 1 and set $B_a(x) = \varphi_a(x - \frac{1}{2})$ and $b_a = D^{m+1} B_a$. It follows that $b_a$ is a $p$-atom with support in the interval $J = (\frac{1}{2} - a/2, \frac{1}{2} + a/2)$.

We have $b_a = \Sigma_2^\infty c_n f_n$ with convergence in $L^2$, where $c_n = \int b_a f_n dt$. It follows that $b_a = \Sigma_2^\infty c_n f_n$ with convergence in $H^p$ and hence

$$a_n = a_n(b_a) = \int b_a f_n dt = (-1)^{m+1} \int B_a D^{m+1} f_n dt.$$

Since $D^{m+1} f_{n_j}$ has a jump of absolute value $\geqq c 2^{j(m+3/2)}$ at the point $\frac{1}{2}$ and is constant on each of the intervals $(\frac{1}{2} - 2^{-j-1}, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2} + 2^{-j-1})$, we obtain

$$|a_{n_j}(b_a)| \geqq c a^{m+1-1/p} 2^{j(m+3/2)} a = c 2^{j(m+3/2)}, \qquad a \leqq 2^{-j}, \quad j \geqq j_0,$$

where $c > 0$.

It is known that there exists a subinterval $I'_n$ of the interval $I_n$ such that

$$|I'_n| \geqq c 2^{-j}$$

and

$$|f_n(x)| \geqq c 2^{j/2}, \qquad x \in I'_n,$$

for some constant $c > 0$ (see the proof of lemma 2 in [6]).

We first assume that $k \geqq 2$. Then the intervals $I_{n_j}$ are disjoint. Invoking (13) and writing $a_n$ instead of $a_n(b_a)$ we obtain

$$C \geqq \left\| \left( \sum_{2}^{\infty} a_n^2 f_n^2 \right)^{1/2} \right\|_p = \left( \int \left( \sum_{2}^{\infty} a_n^2 f_n^2 \right)^{p/2} dt \right)^{1/p}$$

and hence

$$C \geqq \int \left( \sum a_{n_j}^2 f_{n_j}^2 \right)^{p/2} dt \geqq \sum_{\substack{j \geqq j_0 \\ n_j \leqq 1/a}} \int_{I'_{n_j}} \left( \sum a_{n_j}^2 f_{n_j}^2 \right)^{p/2} dt$$

$$\geqq \sum_{\substack{j \geqq j_0 \\ n_j \leqq 1/a}} \int_{I'_{n_j}} (a_{n_j}^2 f_{n_j}^2)^{p/2} dt \geqq c \sum_{\substack{j \geqq j_0 \\ n_j \leqq 1/a}} 2^{j(m+3/2)p} 2^{-j} 2^{jp/2}$$

$$= c \sum_{\substack{j \geqq j_0 \\ n_j \leqq 1/a}} 2^{j(m+2)p} 2^{-j} = c \sum_{\substack{j \geqq j_0 \\ n_j \leqq 1/a}} 1.$$

But the right-hand side tends to infinity as $a \to 0$ and hence we obtain a contradiction.

If $k = 1$ the intervals $I_{n_j}$ are not disjoint, but we can choose $I'_n$ so that $I'_n \subset (1-\delta)I_n$, $\delta > 0$, for every $n$. Then the intervals $I'_{n_{Lj}}$, $j = 1, 2, 3, \cdots$, are disjoint if $L$ is chosen so large that $2^{-L} < \delta$. The above argument now applies with the intervals $I'_{n_j}$ replaced by the intervals $I'_{n_{Lj}}$. It follows that $(f_n)_2^\infty$ is not an unconditional basis of $H^p(I)$ and the proof is complete.

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